The search for “polarized” instantons in the vacuum.

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The new phase of a gauge theory in which the instantons are “polarized”, i.e. have the preferred orientation is discussed. A class of gauge theories with the specific condensates of the scalar fields is considered. In these models there exists an interaction between instantons resulting from one-fermion loop correction. The interaction makes the identical orientation of instantons to be the most probable, permitting one to expect the system to undergo the phase transition into the state with polarized instantons. The existence of this phase is confirmed in the mean-field approximation in which there is the first order phase transition separating the “polarized phase” from the usual non-polarized one. The considered phase can be important for the description of gravity in the framework of the gauge field theory.

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I. INTRODUCTION

In this paper the new nontrivial vacuum state of gauge theory is discussed. In this state instantons are “polarized”, i.e. have the preferred orientation. This polarized state is interesting by itself, but its importance is enhanced by Ref. [1] where a scenario was suggested to describe effects of gravity in the framework of Yang-Mills gauge theory. In this approach space-time is supposed to be basically flat, there are only usual for the Yang-Mills theory fields of spin $0, 1/2$ and 1 on the basic level of the theory. There are neither gravitons nor the dimensional Newton gravitational constant in the basic Lagrangian of the theory. In spite of its “conventional nature” the gauge theory as discussed in [1] can provide a description of the effect of gravity. In recent decades it was supposed that quite new basic physical conceptions, such as supergravity, strings and superstrings, for a review see [2], are vital for quantum gravity.

In Ref. [1] a specific phenomenon based upon nontrivial topological excitations of the gauge field was considered. The most clear topological excitation is the instanton [3]. It has several degrees of freedom: its position, radius and orientation. Generally speaking an instanton may be oriented arbitrarily. It was assumed in [1] that it is possible to construct such the Yang-Mills theory that the instantons in the vacuum state have a preferred direction of their orientation. That means that the probability for any instanton to be oriented along certain direction is greater than along any other direction. This phenomenon can be called “a polarization” of instantons. For formulation of the problem discussed in this paper it is important to recall more precisely the main result of [1]. With this purpose consider $SO(4)$ gauge theory. Then the gauge algebra $so(4)$ is the sum of two $su(2)$ subalgebras: $so(4) = su(2) + su(2)$. It was assumed in [1] that the vacuum state of $SO(4)$ gauge theory possesses the following properties. First, instantons belonging to one $su(2)$ subalgebra of the gauge algebra are polarized. At the same time the antiinstantons belonging to this $su(2)$ subalgebra are not polarized. Second, there is the reversed situation in the other $su(2)$ subalgebra: the antiinstantons belonging to that subalgebra are polarized while the instantons of that subalgebra are not polarized. In other words, in both $su(2)$ subalgebras the situation is nontrivial: in one of them the instantons are polarized while in the other one the antiinstantons are polarized. Note however, that the topological charge in every one $su(2)$ subalgebra might be zero resulting in equal concentrations of instantons and antiinstantons for the given subalgebra.

The concentrations of these polarized instantons of one $su(2)$ subalgebra and polarized antiinstantons of the other $su(2)$ subalgebra are supposed to be finite and equal. One can call this phenomena the “condensate of polarized instantons and antiinstantons” or “Instanton-Antiinstanton Polarization” in the vacuum. Certainly this desired vacuum state can not be achieved in the framework of the pure gauge theory. The theory must acquire some special properties in the scalar and fermion sectors as discussed in detail below. The main result of Ref. [1] is that if the vacuum of the $SO(4)$ gauge theory possesses the condensate discussed then this gauge theory reveals the effect of gravity.

Note first of all that existence of the considered nontrivial phase of the gauge field does not come into contradiction with the gauge invariance of the theory. The gauge invariance forbids those nontrivial phases whose order parameter is not invariant under local gauge transformations [4], [5], [6]. The orientation of instantons cannot be varied by the *local* gauge transformations. This is the well known fact for the pure gauge theory, see Refs. [7], [8]. Therefore the instanton orientation may play the role of an order parameter for the considered nontrivial phase of the gauge theory.

The most “natural” way to look for the condensation of polarized instantons is to find an interaction between instantons which forces any two instantons to have the identical orientation. Then one can expect the system to undergo a phase transition into the state with polarized instantons. An attempt to implement this idea meets two problems. First, in a pure gauge theory instantons do not interact. There is the well-known interaction between instantons and antiinstantons [9], but the interaction between instantons seems to be questionable because there is an exact multi-instanton solution with arbitrary orientation of instantons [10]. Second, along with instanton interaction there must be no interaction between antiinstantons in order to keep the system of antiinstantons disordered, as it is vital for the considered above construction of Ref. [1].

The models considered in this paper include the scalars and fermions in a special way resolving the above mentioned problems. The simplest model is based upon the $SU(2)$ gauge group. The models provide the desirable effective interaction between instantons making their identical orientation the most probable. This result gives a hope that the phase transition into the state with polarized in-

stantons is possible. Moreover, the mean field approximation confirms the existence of the phase with polarized instantons. It is separated from the non-polarized phase by the first-order phase transition.

It is important that the models considered provide the interaction of instantons belonging to the $su(2)$ gauge algebra, but no interaction between the antiinstantons of this algebra. As a result one can expect no condensation of antiinstantons belonging to this $su(2)$ gauge algebra. The models can describe the reversed situation: for a given $SU(2)$ gauge group they can provide the antiinstantons with the interaction while instantons would possess no interaction. Then in this $SU(2)$ gauge group one can expect the antiinstantons to become polarized, while instantons remain non-polarized. This is exactly what is necessary for the discussed above $SO(4)$ gauge group construction, where instantons must be polarized for one $su(2)$ subalgebra and antiinstantons - for the other.

Notice that for $SO(4)$ gauge group the necessary vacuum state of polarized instantons and antiinstantons does not violate the parity conservation law in spite of the fact that there is a clear distinction between instantons and antiinstanton. The polarized instantons belong to the $su(2)$ subalgebra 1, the antiinstantons to the $su(2)$ subalgebra 2. The inversion results in the transformation P: instantons $\rightarrow$ antiinstantons, and antiinstantons $\rightarrow$ instantons. This, however, does not produce the new vacuum state. Really, there is a freedom to arbitrarily choose the two $su(2)$ subalgebras. Simultaneously with inversion one can change the names of the subalgebras, P: subalgebra 1 $\rightarrow$ subalgebra 2, subalgebra 2 $\rightarrow$ subalgebra 1. Under this transformation the polarized instantons remain in the subalgebra 1, the antiinstantons in the subalgebra 2. In the new vacuum state the orientations of instantons and antiinstantons differ from orientations in the initial vacuum. In order to restore the initial orientation it is sufficient to fulfill the global rotation in the isotopic space, which is allowed due to gauge invariance of the theory. The vacuum state remains invariant under thus understood inversion. If all the other properties of the theory are invariant under the transformation: subalgebra 1 $\rightarrow$ subalgebra 2, subalgebra 2 $\rightarrow$ subalgebra 1, then the theory satisfies the parity conservation law. In this paper we deal mainly with $SU(2)$ gauge group. For this gauge group we are looking for the possibility to construct the vacuum with polarized instantons (or antiinstantons). The single $SU(2)$ group with polarized instantons obviously violates the parity conservation law, because inversion results in the transformation of the instantons into antiinstantons. We will keep in mind that the parity would be restored, if one combines two $SU(2)$ groups, having polarized instantons in one of them and polarized antiinstantons in the other.

Consider the main idea. The usual way to govern the properties of the gauge theory is provided by the scalar condensate. It is clear that this condensate itself cannot resolve the puzzle of the different behavior of instantons and antiinstantons. In order to do this let us remember that the instantons and antiinstantons interact differently with right-hand and left-hand fermions. This property manifests itself most strongly for fermion zero-modes: the instantons produce the right-hand fermionic zero-modes, while the antiinstantons give the left-hand zero-modes [11]. Realizing that we can develop the following construction. Let there exist some condensates of the scalars. Let the scalar-fermion interaction has the scalar-pseudoscalar vertex proportional to $1-\gamma_5$. Then the right-hand fermions interact with the scalar condensate, while the left-hand fermions do not. As a result the right-hand zero modes of fermions provide a connection between the scalars and instantons. It permits the scalars to influence upon the instantons in a specific manner. We will see that this influence results in the desirable interaction between the instantons. No similar connection appears between scalars and the antiinstantons because the left-hand fermions do not interact with the scalar condensates.

This construction may be described in the usual terms as a one-fermion-loop correction to the gauge field action. Calculating it we are to consider the interaction of the fermions with the scalar condensate as well as with the gauge field created by several instantons. In order to find the necessary effect one must choose properly the scalar condensates. The condensates create the field V applied to the fermions. It is clear that this field must not commute with the gauge field created by instantons, $[\gamma\nabla, V] \neq 0$, where ∇ is the covariant derivative $\gamma\nabla = \gamma_\mu(\partial_\mu - iA^a_\mu(x)T^a)$, $T^a = \tau^a/2,\ a = 1,2,3$ are the generators of the $SU(2)$ gauge group, and $A^a_\mu(x)$ is the gauge field created by several instantons. Otherwise no connection between the scalar condensates and instantons would appear. If one wishes to consider homogeneous field V, as is usual, then the only way to satisfy this condition is to suppose that the field V depends on the generators of the gauge group: $V \sim \vec{T}\vec{U}(1-\gamma_5)$. As a result we are to introduce into the problem some additional vector $\vec{U}$ to multiply the vector of generators $\vec{T}$. The constant vector $\vec{U}$ not only looks ugly but makes no good, as can be verified. We are to find the vector whose averaged value is zero, but the averaged values of its powers could play a role: $< U^a >= 0,\ \ < U^a U^b >= (1/3)\vec{U}^2\delta_{ab},\ \dots$. The way to do this can only be provided by an additional symmetry. We are to consider some “additional” $SU(2)$ group and identify its generators with the necessary vector $\vec{U}$. There are two ways to introduce this additional symmetry. It may be considered either as a global symmetry or as a local gauge symmetry. For the first case one can visualize group as “a flavor group” or “a group of generations”. This possibility was first discussed in Ref. [12]. For the second case the gauge group considered becomes as big as $SU(2)\times SU(2)$. Both possibilities and shown to provide the system of instantons with the desired properties. There is meanwhile a substantial difference between these two realizations of the model. For the global additional group the gauge field acquire a mass. In contrast, the local realization of the additional symmetry permits to keep the gauge fields for some $SU(2)$ gauge subgroup massless. The latter property is desirable because there must remain the massless gauge field if we wish to construct afterwards the massless gravitons with the help of this field.

Note that there is a clear and interesting analogy between the discussed models and the phenomenon of ferromagnetism. The problem of the phase transition into a state with polarized instantons resembles the transition into a ferromagnetic phase. The instantons play a role very similar to the role of magnetic impurities in the ferromagnetic case. The fermionic zero modes resemble the atomic outer electrons. In the pure gauge theory the

fermionic zero-modes for several instantons remain degenerate in accordance with the index theorem [13], [14]. This resembles the situation when atomic outer electrons are well-localized on the atoms and their energy levels are degenerate. The interaction of right-hand zero-modes with the scalar condensates results in the splitting of the zero modes. Therefore these scalar condensates play the same role as the crystal field which results in the splitting of the atomic energy levels and creation of a conducting band. The splitting makes zero-modes look similar to the electrons in the conducting band. The problem of instanton interaction looks very close to the origin of the exchange integral describing interaction of the magnetic impurities in the ferromagnetic theory. The exchange integral between impurities appears due to their interaction with the electrons of the conducting zone. Similarly the interaction of instantons with the right-hand zero-modes results in the effective interaction between instantons. If some other impurities do not interact with the electrons of the conducting zone then they do not play a role in the ferromagnetic state. Similarly, the antiinstantons which do not interact with the right-hand fermions do not play a role in the considered problem.

The precise results of this paper are the following. The fermionic determinant $\det(F)$ is calculated in the one-loop approximation when fermions interact with instantons as well as with the scalar condensate. If we define

$$\det(F) = \exp(-S_F) \ , \tag{1}$$

then S_F may be considered as an effective action, i.e. a contribution of the fermions to the action for the gauge field. It is found that it describes the interaction of the instantons. For the simplest case of two well separated instantons this interaction is found to be

$$S_F = \frac{f^2\phi^2}{2m^2}\frac{\rho_1^2\rho_2^2}{r^4}\sin^2\gamma \ , \tag{2}$$

where ϕ is the value of the scalar condensate, f is the coupling constant of the scalar-fermion interaction, m is the mass of fermions, ρ_1, ρ_2 are the radii of the instantons and r is their separation, γ is the angle between the directions of instanton orientation. It is assumed in (2) that $\rho_1, \rho_2 \ll r$. Formulas (1), (2) describe the interaction between the instantons. The action (2), first reported in Ref. [12], has a desired minimum at $\gamma = 0$ for the identical orientation of instantons. Note that usually the radiative corrections renormalize physical quantities. In the case considered they provide the new phenomenon: the interaction between the instantons.

The mean field approximation for the ensemble of instantons interacting via action Eq.(2) is considered. This approach confirms the existence of the nontrivial vacuum state with polarized instantons. It reveals also the first-order phase transition which separates the polarized phase from the non-polarized.

In Sections II the model based on $SU(2)$ gauge theory with additional global $SU(2)$ symmetry is considered. The interaction of the instantons is considered in detail in Sections III-V. In Section VI the mean field approximation is developed. In Section VII the $SU(2)\times SU(2)$ gauge theory model is discussed. Throughout the paper we use the set of definitions presented in Section IX. The vital properties of the fermionic zero-modes in the field of several instantons and their role in the instanton-instanton interaction is discussed Sections X,XI.

II. THE $SU(2)$ MODEL

Consider the $SU(2)$ gauge theory. Suppose that there are two generations of fermionic fields in the fundamental representation of this gauge group. The masses of the fermions are supposed to be equal and we will treat them as a doublet in the space of generations. Suppose also that there are scalars in the vector representation of this gauge group. Consider three generations of scalars, e.g. a scalar triplet in the space of generations. Let us introduce the interaction between the scalars and the right-hand fermions described by the Lagrangian

$$\mathcal{L}_{sf}(x) = f\psi_A^+(x)\Phi_i(x)U^i_{AB}\frac{1-\gamma_5}{2}\psi_B(x) \ . \tag{3}$$

Here f is a dimensionless constant of scalar-fermion interaction, $\psi_A(x)$ is the fermion doublet, indexes $A, B = 1, 2$ label the variables in the space of generations, $\Phi_i(x), i = 1, 2, 3$ is the triplet of scalar fields. There is a freedom of choosing the matrixes $U^i = U^i_{AB}, \ i = 1, 2, 3$ describing the coupling between different generations of fermions and scalars. We choose these matrixes to be the triplet of generators of rotations in the space of generations

$$U^i = U^i_{AB} = \sigma^i_{AB}/2, \ i = 1, 2, 3 \ . \tag{4}$$

Note that Euclidean formulation is used, see for example Ref. [15].

Notice that the Lagrangian Eq.(3) obviously violates parity conservation law. It can be restored if, alongside with the considered $SU(2)$ gauge group, we consider the other $SU(2)$ gauge group in which the Lagrangian has the form similar to Eq.(3) but with the positive sign in front of γ_5.

The scalar fields $\Phi_i(x)$ are in the vector representation of the gauge $SU(2)$ group, therefore

$$\Phi_i(x) = \Phi_{i,a}(x)T^a \ , \tag{5}$$

where $T^a = \tau^a/2, \ a = 1, 2, 3$ are the generators of $SU(2)$ gauge transformations. Thus we have nine scalar fields $\Phi_{i,a}(x)$. Suppose now that their nonlinear self-interaction results in developing of the scalar condensate which has the following form

$$(\Phi_{i,a}(x))_{cond} = \phi\delta_{ia} \ , \tag{6}$$

where ϕ is a constant. One can always fulfill this condition choosing appropriately the self-interaction of the scalars. Then it follows from (3),(6) that there appears the field V,

$$V = f\phi\,(\vec{T}\vec{U})\,\frac{1-\gamma_5}{2} \ , \tag{7}$$

which influence upon the right-hand fermions.

Our goal is to calculate the fermionic determinant $\det(-i\gamma\nabla - im - iV)$. It depends on the gauge field $A^a_\mu(x)$, which stands in the covariant derivative

$$\gamma\nabla = \gamma_\mu\nabla_\mu = \gamma_\mu\left(\partial_\mu - iA^a_\mu(x)T^a\right) \ .$$

We wish to evaluate the determinant when the gauge field is created by several instantons. The gauge field and the field V created by the scalar condensate do not commute $[\gamma\nabla, V] \neq 0$, as follows from Eq.(7). This makes the determinant to depend non-trivially on the field V (7). The

determinant is calculated below in the one-loop approximation. This means that the gauge field and the scalar condensate field are considered as the given external fields.

Let us present the fermionic determinant as

$$\det\left(-i\gamma\nabla - im - iV\right) = \quad (8)$$
$$\det\left(-i\gamma\nabla - im\right)\det\left(1 + GV\right) \ , \quad (9)$$

where G denotes the propagator describing behavior of the massive fermions in the gauge field

$$G = (\gamma\nabla + m)^{-1} \ . \quad (10)$$

The first factor $\det\left(-i\gamma\nabla - im\right)$ in (8) describes the known fermion behavior in the pure gauge field. It is not relevant to the effect considered. The important information is contained in the second factor

$$\det\left(F\right) = \det\left(1 + GV\right) = \exp\left(-S_F\right) \ . \quad (11)$$

which we will discuss in detail. The determinant $\det(F)$ may be considered as a contribution of the fermions to the action for the gauge field S_F.

The instantons are known to create the zero-modes of the fermions. These zero-modes play a crucial role in the following calculations. Therefore it is useful to distinguish them in the fermion propagator. With this purpose let us introduce the projection operator P onto the states of zero modes. It satisfies the conditions

$$P^2 = P \ , \ (\gamma\nabla)P = 0, \ \mathrm{Sp}(P) = 2k \ . \quad (12)$$

Here k is the number of instantons. The number of zero-modes in the considered case of two fermionic generations in the fundamental representation of the $SU(2)$ group is $2k$. The propagator (10) may be presented as

$$G \ = G_0 + G_1, \quad (13)$$
$$G_0 = P/m, \quad (14)$$
$$G_1 = (1-P)(\gamma\nabla + m)^{-1}(1-P). \quad (15)$$

To simplify calculations let us consider the instantons so small that the condition

$$m\rho \ll 1 \ , \quad (16)$$

where ρ is the instanton radius is fulfilled. Then the fermionic mass m may be considered as a small parameter and we will put $m = 0$ wherever it is possible. Notice that the instanton separation is not restricted in this consideration. We will return to this point later, see Section V.A.

The propagator of the nonzero-modes for massless case is simplified to be

$$G_1^{(m=0)} = (1-P)(\gamma\nabla)^{-1}(1-P). \quad (17)$$

This propagator may be presented in the convenient form found in Ref. [7]

$$G_1^{(m=0)} = (\gamma\nabla)\frac{1}{\nabla^2}\frac{1+\gamma_5}{2} + \frac{1}{\nabla^2}(\gamma\nabla)\frac{1-\gamma_5}{2} \ . \quad (18)$$

It is clear from (7),(18) that

$$VG_1^{(m=0)}V = 0 \ . \quad (19)$$

Using Eq.(19) we find that only the zero-modes (14) give a contribution to (11), while nonzero-modes are eliminated

$$\det\left(F\right) = \det\left(1 + \frac{1}{m}PV\right) \ . \quad (20)$$

This result greatly simplifies the problem because for the finite number of the instantons the operator PV is presented by a finite matrix.

It is necessary to keep in mind that the scalar condensate (6) results in the creation of the mass for the gauge field through the Higgs mechanism [16]

$$M_V^2 = \frac{3}{4}g^2\phi^2 \ , \quad (21)$$

where g is the gauge coupling constant. We will neglect this mass supposing that the instantons radii ρ are sufficiently small

$$\rho M_V \ll 1 \ . \quad (22)$$

III. THE FERMIONIC DETERMINANT

In order to calculate the determinant (20) we are to consider the matrix V in the subspace of zero-modes where it has a form $V = \langle t, B|V|s, A\rangle$, here the indexes $s, t = 1, \cdots, k$ numerate the zero-modes in one generation of instantons, while the indexes $A, B = 1, 2$ label the generations of zero-modes. According to (84) the wave-functions of the zero-modes have the form

$$\Psi_{s,A} = \Psi_{s,A}(x,\alpha,n) = \frac{1}{\pi}[L_s^+(x)]_{n,n'}\epsilon_{n',\alpha}\omega_A \ . \quad (23)$$

Here the index $\alpha = 1, 2$ is a spinor index of the right-hand spinors, the index $n = 1, 2$ is the isospinor index, $\epsilon_{\alpha,n}$ is the usual 2×2 antisymmetric tensor. $L_s^+(x)$ is a quaternion conjugate to $L_s(x)$, defined in Eq.(79). According to this definition $[L_s^+(x)]_{n,n'} = L_{s,\mu}(x)(\tau_\mu^-)_{n,n'}$. The function ω_A in (23) describes the orientation of a zero-mode in the two-dimensional space of fermionic generations, it is normalized as $\langle\omega_B^+|\omega_A\rangle = \delta_{AB}$.

Using the wave-functions (23) we find the matrix of the operator V (7)

$$\langle t, B|V|s, A\rangle = \sum_{\alpha,n,n'}\int d^4x \Psi_{t,B}^+(x,\alpha,n')(n', B|V|n, A) \Psi_{s,A}(x,\alpha,n) \ , \quad (24)$$

where

$$(n', B|V|n, A) = \frac{f\phi}{2}\vec{\tau}_{n',n}\vec{U}_{BA} \ . \quad (25)$$

After simple algebraic transformations we evaluate from (23),(24),(25) the convenient form for the matrix

$$\langle t, B|V|s, A\rangle = -i\frac{f\phi}{2}\vec{W}_{t,s}\vec{U}_{BA} \ , \quad (26)$$

where

$$\vec{W}_{t,s} = \frac{2}{\pi^2}\mathrm{Re}\int L_t(x)i\vec{\tau}L_s^+(x) \ d^4x \ , \quad (27)$$

$s,t=1,\cdots,k$. According to (27) the matrix $\vec{W}_{s,t}$ is antisymmetric

$$\vec{W}_{t,s}=-\vec{W}_{t,s}\ . \qquad (28)$$

Using (26) we can find the explicit expression for the determinant (20) for a small number k of instantons. For a trivial case of single instanton $k=1$ we find $\det(F)=1$, due to Eq.(28). For two instantons, $k=2$, we find from Eq.(26)

$$PV=\frac{f\phi}{2}\begin{pmatrix}\hat{0} & -i\vec{W}_{1,2}\vec{U}\\ i\vec{W}_{1,2}\vec{U} & \hat{0}\end{pmatrix}\ . \qquad (29)$$

Remember that $\vec{U}$ is the 2×2 matrix in the space of generations. In Eq.(29) $\hat{0}$ is the zero 2×2 matrix in the same space. Diagonalize the matrix in Eq.(29) we find the determinant Eq.(20)

$$\det(F)=\left(1-\frac{1}{4}\zeta^2\vec{W}^2_{1,2}\right)^2\ . \qquad (30)$$

Parameter ζ is

$$\zeta=\frac{f\phi}{2m}\ . \qquad (31)$$

It is instructive to look at Eq.(30) from the point of view of the eigenvalue problem for the operator $\gamma\nabla+V$. In the absence of the perturbation there are four zero modes, which are the solutions of $\gamma\nabla\Psi=0$ with zero eigenvalue. The perturbation V results in the shift of the eigenvalues from the zero value

$$(\gamma\nabla+V)\,\psi=\varepsilon\psi\ . \qquad (32)$$

We restrict our consideration to the subspace of the zero modes, where the operator $\gamma\nabla+V$ is identical to V, $\gamma\nabla+V\equiv PV$, given in Eq.(29). The four-fold degenerate zero eigenvalue is split into two two-fold degenerate eigenvalues ε_+ and ε_-,

$$\varepsilon_\pm=\pm\frac{f\phi}{4}|\vec{W}_{1,2}|\ . \qquad (33)$$

Using them one finds the determinant $\det(F)=(1-\varepsilon_+/m)^2(1-\varepsilon_-/m)^2$ which, of cause, is identical to Eq.(30). This consideration clarifies the physical picture discussed in the introduction: the considered fermionic determinant appears due to splitting of the zero modes.

An important case provides the perturbation theory limit $\zeta^2\vec{W}^2_{t,s}\ll 1$. We find from (30),(11)

$$S_F=\frac{\zeta^2}{2}\vec{W}^2_{1,2}\ . \qquad (34)$$

The presented results show that the fermionic determinant depends on the matrix elements $\vec{W}_{t,s}$ given in Eq.(27). If all instantons have identical orientation $q_t=\rho_t w,\ t=1,\cdots,k$, where ρ_t is radius of t-th instanton, and w satisfying condition $w^+w=1$ describes the instantons orientation, then the matrix elements vanish $\vec{W}_{t,s}=0$. This follows from representation (79) for the functions $L_s(x)$ given in Section VIII, but can be easily understood without calculations. For identical orientation there is no "complex" parameter in the problem. This eliminates the real part of the matrix element from the "complex" operator $i\vec{\tau}$ in Eq.(27). As a result the determinant in this case is trivial $\det(F)=1$. We will see below that in general the determinant is the function of the instanton orientations.

IV. LARGE SEPARATION OF TWO INSTANTONS

Consider the situation when the radii of the two instantons are less then their separation, and therefore the dilute gas approximation is valid. Let us calculate explicitly the fermionic determinant in this case. According to Eq.(26) we are to calculate the matrix element $\vec{W}_{1,2}$ defined in Eq.(27). The functions $L_s(x)$ in the dilute gas approximation are given in (90). Substituting them in Eq.(27) we find

$$\vec{W}_{1,2}=\frac{2}{\pi^2}\mathrm{Re}\int\frac{x-y_1}{|x-y_1|^4}q_1^+ i\vec{\tau}q_2\frac{x^+-y_2^+}{|x-y_2|^4}d^4x\ . \qquad (35)$$

The main contribution to the integral in the above expression comes from the region $|x-y_1|\sim|x-y_2|\sim|y_2-y_1|$ which justifies inequality (89) used to evaluate $L_s(x)$ (90). The simple form obtained for the matrix element (35) is due to the chosen singular gauge which makes the vector potential to be strongly localized on the instantons, see Eq.(72). For our purposes it is sufficient to take the wave functions approximated by (90) in spite of the fact that they satisfy the orthogonal condition (86) with accuracy $\sim 1/r_{12}^2$, which is comparable with the right-hand side of (35). This disadvantage does not manifest itself because we calculate in (35) the matrix element of the "imaginary" operator $i\vec{\tau}$ while in the orthogonal condition (86) the matrix element from "real" unity operator is calculated, see numerical results presented below in Fig.1.

Calculating the integral in (35) we find

$$\vec{W}_{1,2}=\frac{1}{r_{12}^2}(\delta_{\mu\nu}-2v_\mu v_\nu)\mathrm{Re}(\tau_\mu^+q_1^+i\vec{\tau}q_2\tau_\nu^-)\ , \qquad (36)$$

where $r_{12}=|y_2-y_1|$ and $v_\mu=(y_2-y_1)/|y_2-y_1|$. Taking into account identities $\tau_\mu^+q\tau_\mu^-=4\mathrm{Re}(q)$ and $\mathrm{Re}(vqv^+)=|v|^2\mathrm{Re}(q)$ valid for arbitrary quaternions q,v we find

$$\vec{W}_{1,2}=\frac{2}{r_{12}^2}\mathrm{Re}(q_1^+i\vec{\tau}q_2)\ , \qquad (37)$$

$$\vec{W}^2_{1,2}=\frac{4}{r_{12}^4}|q_1|^2|q_2|^2\sin^2\gamma_{12}\ , \qquad (38)$$

where γ_{12} is the angle beween the directions of the orientation of instantons 1 and 2 defined by the identity $\mathrm{Re}(q_1^+q_2)=|q_1||q_2|\cos\gamma_{12}$. Remember that this angle is defined up to mod π because the instantons described by quaternions q and $-q$ give the same field, see Eq.(66).

Substituting Eq.(38) into Eq.(30) we find the determinant for two well separated instantons

$$\det(F)=\left(1-\zeta^2\frac{|q_1|^2|q_2|^2}{r_{12}^4}\sin^2\gamma_{12}\right)^2\ . \qquad (39)$$

We are mainly interested in the region of large separation between instantons. If it is so large that $\zeta^2\vec{W}^2_{t,s}\ll 1$, then we can consider the perturbation theory limit (34). Using (38) we find the effective action describing the interaction of 2 instantons

$$S_F\approx S_F^0=2\zeta^2\frac{|q_1|^2|q_2|^2}{r_{12}^4}\sin^2\gamma_{12}\ . \qquad (40)$$

This formula reproduces Eq.(2) proclaimed in the introduction. The generalization of this result to the

case of several instantons is straightforward, $S_F^0 = \zeta^2 \sum_{s\neq t} |q_s|^2|q_t|^2 \sin^2\gamma_{st}/r_{st}^4$.

Note the peculiarity of $\det(F)$ (39). It possesses a node for specific values of parameters $\zeta, q_1, q_2, r_{12}, \gamma_{12}$. If $\zeta > 1$ then this node may be situated in the region of applicability of the formula. The node has a clear physical meaning. When it happens then there is a "compulsory" creation of the four fermion-antifermion pairs produced by the two considered instantons. To see this consider the instanton contribution to the amplitude of the pair creation. It has a multiple node coming from the determinant and the poles produced by the Green functions describing the propagation of the created fermion pairs in the field of instantons. The node and the poles compensate for each other resulting in the finite amplitude for the physical process. This effect is similar to creation of a pair of massless quarks in the field of an instanton in QCD, see for example [15]. The difference is that in the considered case the fermions possess mass $m > 0$. Note also that the fermionic determinant is finite in the limit $m = 0$. The usual node of the determinant $\det(-i\gamma\nabla - im)$ in the pure gauge field at $m = 0$, see Eq.(8), is compensated for by the pole arising from ζ in Eq.(39).

Let us summarize the conditions under which the obtained results are valid. The radii ρ and separations of instantons r must obviously satisfy condition $\rho < r$ which justifies the dilute gas approximation. Eq.(16), $m\rho < 1$, guarantees the validity of the approximation based on consideration of zero-modes only. Eq.(40) was evaluated from Eq.(39) in the perturbation theory limit $S_F^0 < 1$ valid if the separation is large enough $(\zeta|\sin\gamma|)^{1/2}\rho < r$. The radii of instantons must be small enough to protect them from being cracked by the scalar condensate. This means according to Eq.(22) that $\rho < 1/(g\phi)$.

According to Eq.(39) there is the interaction between the instantons. Eq.(40) shows that it makes their identical orientation $\gamma_{12} = 0$ be the most probable. We did not use perturbation theory over the parameter $\zeta = f\phi/(2m)$. In this sense the problem of interaction between instantons is solved exactly.

V. THE SUPPRESSION OF INSTANTON INTERACTION IN THE ASYMPTOTIC REGIONS

It follows from from Eq.(40) that the effective action for two instantons depends on their separation as $S_F \sim 1/r^4$. It is interesting to consider the action averaged over the positions of instantons. This averaged action has the logarithmic divergence $(S_F)_{\rm av} \sim \int d^4r/r^4$ for small separation $r = 0$ as well as for large separation $r = \infty$. The purpose of this section is to study what is happening in the vicinity of $r \sim 0$ and $r \sim \infty$. It is shown that the interaction of instantons is suppressed in both these regions. The origin of the suppression is different. For small separation the two-instanton interaction itself is strongly suppressed when $r < \rho_1 + \rho_2$. In contrast, there is no suppression in the two-instanton problem for large separation where Eq.(40) gives a correct asymptotic result for the two-instanton problem. The existence of other instantons changes this result drastically: the two-instanton interaction becomes suppressed exponentially in the presence of the other instantons.

A. Small separation of two instantons

Consider first interaction of the two instantons in the region of small separation $r_{12} \sim |q_1|, |q_2|$. Simple analytical results may be obtained in this region if we assume that

$$r_{12}^2 \gg |q_1 q_2 \sin\gamma_{12}| \ . \tag{41}$$

This inequality covers the interesting for us now case of small separation $r_{12}^2 < |q_1 q_2|$ provided the orientation of the two instantons is close enough $|\gamma_{12}| \ll 1$. Remember that we are interesting in the "polarized" instantons, therefore their identical orientation is of particular interest. Inequality (41) covers as well the large separation region, which was considered above. It is shown in Section XI, see Eqs.(117), that the matrix element describing interaction of the two instantons satisfying Eq.(41) is

$$\vec{W}_{1,2} = 2F(\kappa_1,\kappa_2)\frac{{\rm Re}(q_1^+ i\vec{\tau} q_2)}{r_{12}^2} \ , \tag{42}$$

Eq.(42) differs from Eq.(37) by the factor $F(\kappa_1, \kappa_2)$, where $\kappa_i = 2|q_i|/r,\ i = 1, 2$, and $F(\kappa_1, \kappa_2)$ is defined in Eq.(118).

It is convenient to present the action Eq.(34) in the following form

$$S_F = K S_F^0 = 2K\zeta^2 \frac{|q_1|^2|q_2|^2}{r_{12}^4}\sin^2\gamma_{12} \ . \tag{43}$$

Here K is the factor which shows how strongly the action S_F deviates from the asymptotic expression S_F^0 defined in Eq.(40). It follows from Eqs.(34),(42) that for the case under consideration this factor is $K = F^2(\kappa_1, \kappa_2)$. Its behavior as a function of $r_{12}/(|q_1| + |q_2|)$ is illustrated in Fig.1. For large separation $r_{12} \gg |q_1| + |q_2|$ the factor K is close to unity, as is should be in order to reproduce the large separation result Eq.(40). The interesting thing happens for small separation $r_{12} \to 0$. Here K is decreasing very fast. It is easy to find from Eq.(118) the asymptotic behavior $F^2(\kappa_1, \kappa_2) \to r_{12}^4(\ln 1/r_{12} + {\rm const})^2,\ r_{12} \to 0$. Fig.1 shows that this drastic decrease happens starting from $r_{12} \le |q_1| + |q_2|$. Thus the interaction of the two instantons is strongly suppressed for small separation. According to Eq.(43) the averaged action in the small-separation region behaves as $(S_F)_{\rm av} \sim \int K dr/r$. The later integral is convergent, the main contribution to it gives the region in which $r_{12} \ge |q_1| + |q_2|$. The other interesting feature illustrated by Fig.1 is the fact that $F^2(\kappa_1, \kappa_2)$ is almost independent on the ratio of the instanton radiuses $|q_1|/|q_2|$ thus depending mainly on $r_{12}/(|q_1| + |q_2|)$.

Up to now we have discussed the region covered by Eq.(41), which gives the strong restriction on γ_{12} for small separation of instantons. This was done in order to develop analytical calculations as far as it is possible. The general case of small separation and arbitrary orientation of two instantons can be handled by direct numerical calculations. The useful representation for the matrix element $\vec{W}_{1,2}$ is given in Section XI in Eq.(110). In general case the factor K defined by Eq.(43) is a function of three variables $K = K(\kappa_1, \kappa_2, \gamma_{12})$. The results of numerical calculations for K based on Eqs.(110),(43) are also presented in Fig.1. The suppression of configurations with the small separation $r_{12} \le |q_1| + |q_2|$ is the general feature, it is valid for any orientations and radii of instantons. Notice the very smooth dependence of K

on γ_{12}. It manifests itself mainly in the region of small separation, which is suppressed. Thus the considered factor K with reasonably good accuracy may be considered as a function of only one variable $r_{12}/(|q_1|+|q_2|)$, $K \approx F^2\left((|q_1|+|q_2|)/r_{12},(|q_1|+|q_2|)/r_{12}\right)$, where the function $F\left(\kappa_1,\kappa_2\right)$ is defined in Eq.(118).

We conclude: the interaction of the instantons is suppressed by the factor K when their separation is small.

B. Large separation of two instantons in the presence of other instantons

Consider two instantons 1 and 2 separated by large distance $r_{12} \gg |q_1|,|q_2|$. Our goal is to examine what is happening with their interaction if there are other instantons in the vicinity of either instanton 1 or 2. We know that the most important role in the problem play the fermionic zero-modes. We assumed in the previous consideration that the scalar field is weak enough, it does not seriously change the behavior of the zero-modes. The basis for this assumption provides the inequality $f\phi \ll 1/\rho$. When one considers very large separation of instantons, then the zero-mode is to be considered far outside of the instanton radius where even the weak scalar field can drastically change the zero-mode behavior. Note that in the latter consideration we continue to call the zero-modes those functions which originate from zero-modes of the pure gauge theory. The scalar condensate influence upon them, shifting their eigenvalues from the zero value but this shift is supposed be small.

Calculating the fermionic determinant Eq.(8) one is to face an eigenvalue problem Eq.(32), $(\gamma\nabla+V)\,\psi=\varepsilon\psi$. It is convenient for our purposes to re-wright it as

$$\psi(x)=-\int G(x-x',\varepsilon)\gamma_\mu A_\mu(x')\psi(x')d^4x' \ , \tag{44}$$

where $A_\mu(x) = -iA^a_\mu(x)\tau^a/2$, and $G(x,\varepsilon)$ is the Green function whose kernel $G(\varepsilon)$ is

$$G(\varepsilon)=\frac{1}{\gamma\partial+V-\varepsilon}=\frac{\gamma\partial-V_0(1+\gamma_5)/2+\varepsilon}{\partial^2+\varepsilon(V_0-\varepsilon)} \ , \tag{45}$$

where $V_0 = f\phi\vec{T}\vec{U}$. Eq.(44) is convenient for the asymptotic expansion because in the singular gauge the vector-potential $A(x')$ is localized on the instanton. Consider the zero mode $\psi(x)=\psi_s(x)$ localized on the instanton s. Then the integration in Eq.(44) is also localized in the vicinity of the instanton s. As a result we find from Eq.(44) the asymptotic

$$\psi_s(x)\to G(x-y_s,\varepsilon)\lambda_s \ , \quad |x-y_s|\to\infty \ , \tag{46}$$

where $\lambda_s=-\int\gamma_\mu A_\mu(x')\psi_s(x')d^4x'$. It is easy to find the constant λ_s. Really, we assume that the scalar field $f\phi$ is so weak that the zero-mode is not influenced by the condensate in the instanton vicinity. Therefore λ_s may be found using the pure zero-modes, non-affected by the scalar condensate. The pure zero-mode $\Psi_s(x)$ satisfies the asymptotic condition similar to Eq.(46) with the Green function being $G_0(x)=\gamma\partial/\partial^2=\gamma x/(2\pi^2x^4)$. Comparing this with Eqs.(84),(90) one finds that λ_s is the left-hand Dirac spinor $\lambda_s=\lambda^L_{s,A}(\alpha,n)=2\pi q_{s,nn'}\epsilon_{n'\alpha}\omega_A$.

We see from Eq.(45) that the asymptotic behavior very strongly depends on the eigenvalue ε. For the pure zero-mode $\varepsilon=0$ there is only the possibility for the power-type decrease of the zero-mode with separation. Even the small shift of the eigenvalue from the zero value changes the situation drastically. If $\varepsilon\neq0$, then there appears the constant $\varepsilon(V_0-\varepsilon)$ in the denominator in the Green function Eq.(45) resulting in the exponential decrease of the zero-mode Eq.(44).

This consideration justifies the results of Section IV for the two-instanton problem. Really, in this case there certainly appear the nonzero eigenvalues, but their magnitude decreases with separation as $\sim 1/r^4$. As a result one can neglect $\varepsilon\approx0$ in the denominator of the Green function, thus resulting in the power type decrease of the zero-mode.

The situation changes if there are other instantons. Let for example there is the instanton $1'$ in the vicinity of the instanton 1 and the instanton $2'$ in the vicinity of the instanton 2. Let their separations satisfy $r_{11'},r_{22'}\ll r_{12}$. Then the eigenvalues of the zero-modes centered on the instantons $1,1'$ are influenced mainly by their mutual interaction, being independent on the large distance r_{12}. The same happens with eigenvalues of the zero-modes centered on the instantons $2,2'$. This results in the exponential decrease of the zero-modes. The zero modes centered on instantons $1,1'$ or on $2,2'$ decrease as

$$\psi_s\sim\exp(-M_s|x-x_s|), \quad s=1,2 \ . \tag{47}$$

Here $M_s=(\varepsilon_s^2-\varepsilon_sV_0)^{1/2}$. According to Eq.(45) it depends on the splitting $\varepsilon_s=\varepsilon_{s\pm}$ which can be found from Eqs.(33),(37), $\varepsilon_{s\pm}\approx\pm f\phi|q_s||q_{s'}|\sin\gamma_{ss'}/(2r^2_{ss'})$. M_s depends as well on the eigenvalues $f\phi/4,-3f\phi/4$ of the operator $V_0=f\phi\vec{T}\vec{U}$. The exponential decrease of the zero modes with separation results in the decrease of the matrix element $\vec{W}_{12}$ responsible for the interaction between instantons 1 and 2. This consideration shows that the interaction of the instantons $1,2$ can be described by the power law Eq.(40) only if

$$r_{12}<r_{\max}=\frac{1}{f\phi} \ . \tag{48}$$

We estimate here the splitting of the eigenvalues of instantons $1,1'$ and $2,2'$ as $|\varepsilon_{s\pm}|\sim f\phi$.

We conclude, if $r_{12}>r_{\max}$ then the interaction between instantons decreases exponentially with separation r_{12}.

VI. THE MEAN FIELD APPROXIMATION

Let us examine the possibility of the phase transition into the state with polarized instantons in the mean field approximation. Consider the ensemble of instantons. According to Eq.(40) their interaction is described by the action

$$S_F=\zeta^2{\sum_{s\neq t}}'\frac{|q_s|^2|q_t|^2}{r^4_{st}}\left(1-(n_{s,\mu}n_{t,\mu})^2\right) \ . \tag{49}$$

We took here into account that $\sin^2\gamma_{st}=1-(n_{s,\mu}n_{t,\mu})^2$, where $n_{s,\mu}=q_{s,\mu}/|q_s|$ is the unit vector characterizing the orientation of the instanton s. It is shown in Section V.A that the interaction of instantons is suppressed for

small separation $r_{st} < |q_s| + |q_t|$. In Section V.B it is shown that the interaction is suppressed for large separation $r_{st} > r_{\max}$ as well, here $r_{\max}$ is estimated in Eq.(48) $r_{\max} \sim 1/(f\phi)$. The suppression of the interaction in the asymptotic regions is taken into account in Eq.(49) by neglecting the contribution of the suppressed regions. The prime over summation in Eq.(49) reminds one of this fact.

The corresponding statistical sum for the ensemble of instantons is

$$Z = \sum w \exp(-S_F) \ , \tag{50}$$

where summation runs over the number of instantons, their positions, radiuses and orientations. The summation includes also the fluctuations above the pure instanton picture. The function w in Eq.(50) is the probability to find the noninteracting ensemble of instantons in some particular state.

In the mean field approximation one instanton is to be considered in the effective field created by the other instantons. The corresponding action is to be obtained averaging the action (49) over the positions, radiuses and orientations of other instantons. Moreover, we are mainly interested in one and only one degree of freedom for the considered instanton, its orientation. Therefore it is natural to fulfill the averaging over the radius of the considered instanton as well. As a result one finds the effective action in the mean-field approximation as the function of the instanton orientation n

$$S_{\mathrm{mf}}(n) = A n_\mu n_\nu < n_\mu n_\nu > \ , \quad \text{where} \quad A = 2\zeta^2 \frac{\rho_0^4}{R^4} \ . \tag{51}$$

Here $\rho_0^2 =< \rho^2 >$ is the averaged square of the instanton radius. Remember that there exists the scalar condensate which makes this radius to be finite, $\rho_0 \sim 1/M_V \sim 1/(g\phi)$, see Eq.(21). The quantity $< n_\mu n_\nu >$ describes the averaged orientation of the instantons in the vacuum. For non-polarized vacuum, when there is no preferred orientation, $< n_\mu n_\nu >= \delta_{\mu\nu}/4$. The factor $1/R^4$ describes the separation of the instanton from other ones summed over their positions

$$\frac{1}{R^4} = N \int d^4 r \frac{1}{r^4} = 2\pi^2 N \ln \frac{r_{\max}}{r_{\min}} \ , \tag{52}$$

where N is the instanton concentration. We can put $r_{\min} \approx 2\rho_0 \sim 1/(g\phi)$, because for shorter separation instantons do not interact, while $r_{\max}$ according to Eq.(48) is $r_{\max} \approx 1/(f\phi)$. Thus $\ln(r_{\max}/r_{\min}) \approx \ln(g/f)$. Using Eq.(52) one can present the parameter A in the form

$$A = 8\zeta^2 v_0 N L \equiv 8B \ , \quad \text{where} \quad B = \zeta^2 v_0 N L \ . \tag{53}$$

Here $v_0 = \pi^2 \rho_0^4/2$ is the volume occupied by the instanton with the radius ρ_0 and $L = \ln(r_{\max}/r_{\min}) \sim \ln(g/f)$.

The probability to find the instanton with orientation n in the mean-field approximation is $w_{\mathrm{mf}}(n) = \exp S_{\mathrm{mf}}(n)$. The self-consistency condition for the averaged orientation $< n_\mu n_\nu >$ is

$$< n_\mu n_\nu >= \frac{\int n_\mu n_\nu \exp(A < n_\sigma n_\tau > n_\sigma n_\tau) dn}{\int \exp(A < n_\sigma n_\tau > n_\sigma n_\tau) dn} \ . \tag{54}$$

The integration here covers the sphere $n_\mu n_\mu = 1$. The fact of polarization of instantons means that there is at least one eigenvalue of the matrix $< n_\mu n_\nu >$ which exceeds $1/4$. In order to find such an eigenvalue it is sufficient to use for this matrix the anzats

$$< n_\mu n_\nu >= \alpha^2 \delta_{\mu 4} \delta_{\nu 4} + \frac{1-\alpha^2}{4} \delta_{\mu\nu} \ , \tag{55}$$

in which it is supposed that the possible preferred orientation takes place along the 4-axes. Substituting Eq.(55) into Eq.(54) one finds the equation for α^2

$$\frac{z}{8B} = f(z), \quad \text{where} \quad z = 8B\alpha^2 \ , \tag{56}$$

and the function $f(z)$ is

$$f(z) = \frac{4 \int n_4^2 \exp(z n_4^2) dn}{3 \int \exp(z n_4^2) dn} - \frac{1}{3} = \frac{1}{3} \frac{I_1(z/2) - I_2(z/2)}{I_0(z/2) - I_1 \ (z/2)} \ . \tag{57}$$

Here $I_m(x), \ m = 0,1,2$ are the modified Bessel functions. The function $f(z)$ has the properties: $f(z) \to 1, \ z \to \infty, \ f'(0) = 1/12 \approx 0.083 < \max(f(z)/z) \approx 0.110$. The later inequality indicates that $f''(z)$ changes the sigh. Fig.2 shows $f(z)$ as well as the left-hand side of Eq.(56) for different parameters B. The nontrivial solutions of Eq.(56) exist if

$$B = \zeta^2 v_0 N L \geq B_c = 1.14 \ , \tag{58}$$

It is clear that $v_0 N$ is to be small, $v_0 N < 1$, otherwise the instanton overlapping would diminish their interaction. It is natural to put aside the peculiar possibility that $\ln(g/f)$ may be considered as a large parameter. Then the only way to satisfy inequality in Eq.(58) is to have $\zeta > 1$. It is important that this region of ζ can be considered in the framework of the formalism presented above. Really, the only place where the perturbation theory over ζ was used is the evaluation of the action Eq.(40). One should use here the perturbation theory over the parameter $\zeta^2 \rho_1^2 \rho_2^2 \sin^2 \gamma_{12}/r_{12}^4 < 1$, which can be small even for large $\zeta > 1$.

We see from Fig.2 that if $B > B_c' = 3/2$ then the nontrivial solution is unique. In the region $B_c < B < B_c'$ there are two non-zero solutions. One of them is unstable in the following sense. The fluctuation of the instanton polarization in the partition function described by the right-hand side of Eq.(54) results in the stronger fluctuation of the averaged polarization in the left-hand side. The existence of nontrivial solutions means that there appears the state with polarized instantons. The fact that there are two solutions in the vicinity of the critical value B_c indicates that the nontrivial phase appears as a result of the first-order phase transition.

Notice that the action Eq.(49) depends on the scalar products of the vectors $n_{s,\mu}$. This feature makes it look similar to the classical Heisenberg model for ferromagnetics. The only distinction is the second power of the scalar product $(n_{s,\mu} n_{t,\mu})^2$ in Eq.(49). There is a clear topological reason for this peculiarity. An instanton describes the gauge field which belongs to the adjoint representation of the gauge $SU(2)$ group. The different $SU(2)$ matrixes $\pm n_\mu \tau_\mu^+$ have the same representatives in the adjoint representation. In this sense $n_\mu \equiv -n_\mu$. Therefore all the quantities describing the instanton can depend only on even powers of n_μ. In the mean field approximation this second power $(n_{s,\mu} n_{t,\mu})^2$ strongly manifests itself. It results in the change of sign of $f''(z)$ which gives two nontrivial solutions for $B_c < B < B_c'$, and consequently the first order phase transition.

Let us discuss shortly the most important assumptions which are made in this section.
1.We consider the contribution of the instantons to the statistical sum. The quantum fluctuations in the vacuum above the instantons are divided into two parts. One of them is the one-loop contribution of the fermions. These particular fluctuations are of grate importance because they provide the interaction between instanton making their identical orientation be more probable. They are taken into account in Eq.(50) explicitly through the action S_F. Notice that the role of these fluctuations is enhanced by the parameter $\zeta > 1$. All the rest fluctuations present themselves in Eq.(50) implicitly in the function w. In the mean field approximation all these fluctuations manifest themselves through the averaged values v_0, N, L.
2.We assumed that the picture based on the dilute gas approximation for the instantons is valid. There are two reasons in favor of this assumption. First, there is the scalar condensate in the model which suppresses the instantons of large radius. Second, if there is an overlapping between some instantons violating the picture of the dilute gas approximation, then still we know that the interaction of these overlapping instantons is suppressed. Therefore only the configurations with non-overlapping instantons should play an important role.
3. The instantons are taken into account explicitly, the antiinstantons are neglected. The reason in favor of this approach gives a fact that in the considered model the instantons interact and the antiinstantons do not. There is however an interesting problem: there exists the instanton-antiinstanton interaction [9]. It is necessary therefore to examine the role of this interaction in the considered vacuum. This problem remains outside the scope of the present paper.
4.The interaction of instantons is considered as a sum of two-bodies interactions. The virial corrections caused by simultaneous interaction of three or more instantons are not considered. This approximation is consistent with the dilute gas approximation.
5.The mean field approximation for the ensemble of instantons is implemented in the most simple way. Only one degree of freedom - the polarization of instantons is taken into account in the effective action. All other degrees of freedom are averaged out of the action.

In conclusion, the mean field approximation confirms the existence of the vacuum with polarized instantons. The transition to the polarized state in this approximation is the first-order phase transition.

VII. THE MODELS WITH $SU(2) \times SU(2) \subset G$ GAUGE SYMMETRY

Consider the gauge group G which is big enough to include the product of two $SU(2)$ groups: $G_1 \times G_2 \subset G$, where $G_1 = G_2 = SU(2)$. Then the idea of the model developed in Section II may be presented in the other useful way. It is sufficient to consider in this case one generation of fermions which belongs to the fundamental representation of both G_1 and G_2. Let $\Phi(x)$ be a scalar field in the vector representations of G_1 and G_2

$$\Phi(x) = \Phi_{a,b}(x) T_1^a T_2^b \ , \tag{59}$$

here $\vec{T}_1, \vec{T}_2$ are the generators of G_1 and G_2. Let us introduce the scalar-fermion interaction described by the Lagrangian

$$\mathcal{L}_{sf}(x) = f \Psi^+(x) \Phi(x) \frac{1-\gamma_5}{2} \Psi(x) \ . \tag{60}$$

Suppose that the field $\Phi(x)$ develops the condensate

$$(\Phi(x))_{cond} = \phi \vec{T}_1 \vec{T}_2 \ . \tag{61}$$

Then there appears the field V which influence upon the right-hand fermions

$$V = f \phi \vec{T}_1 \vec{T}_2 \frac{1-\gamma_5}{2} \ . \tag{62}$$

Consider now several instantons whose gauge field is transformed under G_1. Then we can identify vector $\vec{T}_1$ in Eq.(62) with $\vec{T}$ in Eq.(7) and vector $\vec{T}_2$ with $\vec{U}$. The fact that $\vec{T}_2$ generates the gauge group does not play a role here because the instantons under consideration belong to G_1, they are not transformed under G_2. Therefore the fermion determinant for the considered case is identical to the determinant discussed in the previous sections, and Eqs.(30), (34),(39) remain valid for the considered model as well. As we know there appears the interaction between instantons with tendency to make their orientation be identical, see (40). The same is true for the instantons belonging to G_2.

Up to this point there is the close similarity beween the considered model and the one discussed previously in Section II. But there is an important distinction. The scalar condensate (61) is invariant under the $SU(2)$ gauge group generated by $\vec{T} = \vec{T}_1 + \vec{T}_2$. Therefore the three vector fields transformed under this group acquire no mass from the scalar condensate

$$M_V = 0 \ . \tag{63}$$

This is in contrast to the model of Section II where all the gauge fields possess the mass (21).

VIII. CONCLUSION

It is shown that the considered models provide the strong interaction between instantons making their identical orientation to be more preferable. At the same time there is no such interaction between the antiinstantons. One can easily reverse the situation changing the sign in front of γ_5 in Eq.(7) or in Eq.(60). Then antiinstantons interact and instantons do not. The interaction appears from one-fermion-loop correction to the gauge action.

We considered two models. One based upon the $SU(2)$ gauge group. The other model is based upon larger gauge group $G_1 \times G_2 \subset G,\ G_1 = G_2 = SU(2)$. These two models agree in supplying the instantons with the interaction. The advantage of the model based upon the wider group is the fact that it preserves the massless gauge fields. These fields are transformed under the gauge group $SU(2)$ overlapping with the groups $G_1,\ G_2$ in which the interaction of instantons takes place. If the condensate of polarized instantons is developed in G_1 and (or) G_2 then we will enjoy the possibility to consider the interaction of the massless gauge fields with that condensate. This is exactly what is necessary for the construction of Ref. [1].

It is shown that the mean field approximation confirms the existence of the phase with polarized instantons which

is separated from the non-polarized phase by the first-order phase transition.

The obtained results give a hope that there exists the state with polarized instantons in gauge theory.

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IX. APPENDIX A

In this paper the following definitions for isospinor and spinor matrixes are used

$$\tau^{\pm}_{\mu} = (\pm i\vec{\tau}, 1), \quad \sigma^{\pm}_{\mu} = (\pm i\vec{\sigma}, 1) \ .$$

The t'Hooft symbols are

$$\begin{aligned}\eta^{a}_{\mu\nu} &= \epsilon_{\mu\nu a4} + \delta_{a\mu}\delta_{4\nu} - \delta_{a\nu}\delta_{4\mu} \ , \\ \bar{\eta}^{a}_{\mu\nu} &= \epsilon_{\mu\nu a4} - \delta_{a\mu}\delta_{4\nu} + \delta_{a\nu}\delta_{4\mu} \ .\end{aligned}$$

The Dirac matrixes γ_μ in the Eucledean space are $\gamma_\mu = (-i\vec{\gamma}^M, \gamma^M_0)$, where γ^M_μ are the usual Dirac matrixes in Minkowsky space. The Eucledean matrixes satisfy the condition $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. In the spinor representation they have the form

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma^+_\mu \\ \sigma^-_\mu & 0 \end{pmatrix}, \ \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

Consider the k-instanton general solution [10], for the revew see [20]. Let us introduce a quaternion as $q = q_\mu \tau^+_\mu$, where $q_\mu, \mu = 1, \cdots, 4$ is an arbitrary vector. In this notation $q^+ = q_\mu \tau^-_\mu$. Consider the $(k+1)\times k$ matrix $M_{s,t}(x), s = 1, \cdots, k+1, \ t = 1, \cdots, k$ which has quaternionic matrix elements. $M(x)$ is supposed to be a linear function of the quaternion of coordinates $x = x_\mu \tau^+_\mu$

$$M(x) = B - Cx \ ,$$

where B and C are x-independent $(k+1)\times k$ quaternionic matrixes. They must be chosen so that the condition

$$M^+(x)M(x) = R(x) \qquad (64)$$

be fulfilled for any x. Here $R(x)$ is a non-degenerate $k\times k$ real matrix. The matrix C can be chosen to be

$$C_{1t} = 0, \quad C_{1+s,t} = \delta_{st}, \ s, t = 1, \cdots, k \ . \qquad (65)$$

Having $M(x)$ one can find the $k+1$ quaternionic vector $N(x)$ which satisfies the equations

$$M^+(x)N(x) = 0, \ N^+(x)N(x) = 1 \ .$$

Then the vector-potential defined in the quaternion representation as

$$A_\mu(x) = -iA^a_\mu(x)\tau^a/2 = N^+(x)\partial_\mu N(x)$$

results in the general self-dual gauge field for k instantons $F_{\mu\nu}(x) = -iF^a_{\mu\nu}(x)\tau^a/2 = [\nabla_\mu, \nabla_\nu]$, where $\nabla_\mu = \partial_\mu + A^+_\mu(x)$. For single instanton

$$M(x) = \begin{pmatrix} q \\ y - x \end{pmatrix}, \ N(x) = \frac{1}{\rho_0(x)} \begin{pmatrix} 1 \\ \frac{x-y}{|x-y|^2} q^+ \end{pmatrix} u \ ,$$

where $\rho^2_0(x) = 1 + \rho^2/|x-y|^2$. The quaternion q describes the instanton orientation, which may be considered as a unit four-dimensional vector $n = q/|q|$, and the instanton radius $\rho = |q|$, the position of the instanton is given by the quaternion y. The quaternion u satisfying condition $|u|^2 = 1$ describes the freedom of the gauge transformations. The condition $u = 1$ corresponds to the singular gauge [11] in which the vector potential has the form

$$A_\mu(x) = -\frac{i}{2\rho_0(x)}\eta^a_{\mu\nu}\frac{q\tau^a q^+(x-y)_\nu}{|x-y|^4} \ . \qquad (66)$$

Similar simple formulas are valid for several instantons if they are well-separated $\rho \ll r$, where ρ is the radius of an instanton and r is a separation of this instanton from the others. Then the dilute gas approximation considered in Ref. [8] is valid. In this approximation the first row of the matrix $M(x)$ is given by k constant quaternions

$$M_{1,t}(x) = q_t, \ t = 1, \cdots, k \ . \qquad (67)$$

while the remaining matrix elements may be approximated as

$$M_{1+s,t}(x) = \delta_{st}(y_t - x), \ s, t = 1, \cdots, k \ . \qquad (68)$$

Here q_t describes the radius and orientation of the t-th instanton, y_t is the position of this instanton. The components of the vector $N(x)$ are

$$N_1(x) = \frac{1}{\sqrt{\rho_0(x)}}u \ , \qquad (69)$$

$$N_{1+t}(x) = \frac{1}{\sqrt{\rho_0(x)}}\frac{x-y_t}{|x-y_t|^2}q^+_t u, \ t = 1, \cdots, k, \qquad (70)$$

where

$$\rho_0(x) = 1 + \sum_{t=1}^{k}\frac{|q_t|^2}{|x-y_t|^2} \ . \qquad (71)$$

The vector potential for k well separated instantons in the singular gauge $u = 1$ has the form

$$A_\mu(x) = -\frac{i}{2\rho_0(x)}\eta^a_{\mu\nu}\sum_{t=1}^{k}\frac{q_t\tau^a q^+_t(x-y_t)_\nu}{|x-y_t|^4} \ . \qquad (72)$$

There is a particular case [17], [18], [19] which is interesting for our purposes: the identical orientation of the instantons

$$q_t = \rho_t w \ . \qquad (73)$$

Here ρ_t is the real radius of t-th instanton and the quaternion $w, \ w^+w = 1$, describes the common orientation of all the instantons. It is known [8] that in this case Eqs.(67) – (72) are valid for any separation between instantons.

X. APPENDIX B. THE FERMIONIC ZERO-MODES

The purpose of this section is to evaluate the convenient form for fermionic zero-modes in the field of several instantons. The projection operator P onto the space of fermionic zero-modes was obtained in [7] in the following form

$$P = \left(1 - (\gamma\nabla)\frac{1}{\nabla^2}(\gamma\nabla)\right)\frac{1-\gamma_5}{2} \ . \qquad (74)$$

The following simple arguments can be used to verify it. It is easy to check out directly that Eqs.(12) $P^2 = P,\ (\gamma\nabla)P = 0$ are fulfilled and therefore P gives a projection into the space of zero-modes. In order to prove that it gives the projection on the hole space of zero-modes it is sufficient to show that $Sp(P)$ is equal to the number of zero-modes. It follows from (74) that

$$\mathrm{Sp}(P) = \mathrm{Sp}\left(\frac{1-\gamma_5}{2} - (\gamma\nabla)^2\frac{1}{\nabla^2}\frac{1+\gamma_5}{2}\right) \ ,$$

and therefore

$$\mathrm{Sp}(P) = \mathrm{Sp}(-\gamma_5) \ . \qquad (75)$$

Every zero-mode gives a unity contribution to the right-hand side. The nonzero-modes do not contribute to it. Indeed, for any nonzero-mode $\psi_\lambda(x)$ satisfying the equation

$$-\,i(\gamma\nabla)\psi_\lambda(x) = \lambda\psi_\lambda(x), \qquad (76)$$

with $|\lambda| > 0$ the function $\psi_{-\lambda}(x) = \gamma_5\psi_\lambda(x)$ satisfies $-i(\gamma\nabla)\psi_{-\lambda}(x) = -\lambda\psi_{-\lambda}(x)$ and is, therefore, orthogonal to $\psi_\lambda(x)$. Thus the right-hand side of Eq.(75) is equal to the number of zero modes. This concludes the verification of the fact that P defined in (74) is the projection operator on the zero-modes space.

The Green function $-1/\nabla^2$ describing propagation of scalars in the field of several instantons was found in Refs. [7], [8]. For the considered case of isospin-1/2 it reads

$$D(x,y) = -\frac{1}{\nabla^2} = \frac{1}{4\pi^2}\frac{N^+(x)N(y)}{(x-y)^2} \ . \qquad (77)$$

Substituting this expression into Eq.(74), after lengthy but conventional calculations presented below, see Eqs.(91)-(102), it is possible to present the kernel $P(x,y)$ of the projection operator in the following simple form

$$P(x,y) = \frac{1}{2\pi^2}L^+(x)C^+C\frac{1-\gamma_5}{2}(1-\vec\sigma\vec\tau)L(y) \ . \qquad (78)$$

Here $L(x)$ is defined as

$$L(x) = R^{-1}(x)C^+N(x) \ . \qquad (79)$$

The quantities $M(x), N(x), B, C, R(x)$ constitute the ADHM construction Ref. [10] defined in Section VII. Note that $L(x)$ is the vector with quaternionic components $L(x) = (L_s(x), s = 1,\cdots,k),\ L_s(x) = L_{s,\mu}(x)\tau^+_\mu$, where k is the number of instantons.

The kernel of the projection operator satisfies the equation $(\gamma\nabla)P(x,y) = 0$ for arbitrary y and the validity of this equation does not depend on specific properties of $L(y)$ in the representation (78). Therefore we can obtain the expression for a zero-mode wave function by replacing $L(y)$ with arbitrary x-independent spinor-isospinor χ. If we define

$$\psi = \sigma^+_\mu\tau^+_\mu(1-\gamma_5)\chi \ , \qquad (80)$$

then we find from Eq.(78) the wave functions of the fermionic zero modes

$$\Psi_s(x) = \frac{\sqrt{2}}{\pi}L^+_s(x)\psi,\ s = 1,\cdots,k \ . \qquad (81)$$

It follows from Eq.(80) that x-independent spinor-isospinor ψ satisfies the following conditions

$$\gamma_5\psi = -\psi,\ \vec\sigma\vec\tau\psi = -3\psi \ . \qquad (82)$$

Let us define $\alpha = 1,2$ the spinor index of the right-hand spinor and $n = 1,2$ the isospinor index. Then Eqs.(82) result in

$$\psi = \psi^R_{\alpha,n} = \frac{1}{\sqrt{2}}\epsilon_{n,\alpha} \ , \qquad (83)$$

where ψ^R is the right-hand spinor-isospinor defined by this equation and $\epsilon_{n,\alpha}$ is the usual antisymmetric tensor with $\epsilon_{1,2} = 1$. Using Eq.(83) one can present the zero-modes given in Eq.(81) in the more detailed form

$$\Psi_s(x) = \Psi_s(x,\alpha,n) = \frac{1}{\pi}\left[L^+(x)\right]_{n,n'}\epsilon_{n',\alpha} \ . \qquad (84)$$

The spinor ψ is normalized in Eq.(83) as $\langle\psi^+|\psi\rangle = 1$. This results in the usual normalization conditions for the functions $\Psi_s(x)$

$$\langle\Psi^+_t|\Psi_s\rangle = \delta_{st} \ . \qquad (85)$$

The easiest way to check out these normalization conditions provides the equation $P^2 = P$ for the projection operator. For the kernel $P(x,y)$ (78) it reads

$$\begin{aligned}&(2\pi^2)^{-2}L^+(x)C^+C\sigma^+_\mu\tau^+_\mu\,\cdot\\ &\int d^4z L(z)L^+(z)\sigma^+_\nu\tau^+_\nu C^+CL(y)[(1-\gamma_5)/2] =\\ &(2\pi^2)^{-1}L^+(x)\sigma^+_\mu\tau^+_\mu C^+CL(y)[(1-\gamma_5)/2] \ .\end{aligned}$$

To satisfy this condition $L(x)$ must obey

$$\begin{aligned}&(2\pi^2)^{-1}(\sigma^+_\mu\sigma^+_\nu)\left(\tau^+_\mu\int L(z)L^+(z)d^4z\ \tau^+_\nu\right) =\\ &(C^+C)^{-1}\sigma^+_\mu\tau^+_\nu \ .\end{aligned}$$

Using identity $(\sigma^+_\mu\sigma^+_\nu)(\tau^+_\mu q\tau^+_\nu) = 4\sigma^+_\mu\tau^+_\mu\mathrm{Re}(q)$, which is valid for any quaternion q we find

$$\frac{2}{\pi^2}\mathrm{Re}\int L^+_t(x)L_s(x)\ d^4x = (C^+C)^{-1}_{st} = \delta_{st} \ . \qquad (86)$$

Here we choose the matrix C to satisfy identity $(C^+C)_{st} = \delta_{st}$, which can be fulfilled for general k-instanton solution, see (65). The normalization (85) follows from (86).

Formula (81) is valid for one generation of fermions in the fundamental representation of $SU(2)$ gauge group. Generalization to the case of several generations of fermions is strait-forward.

The formula becomes much simpler if the dilute gas approximation is valid. It follows from Eqs.(64),(67),(68) that in this case the matrix $R(x)$ simplifies to be

$$R_{st}(x) = \delta_{st}\left(|x-y_t|^2 + |q_t|^2\right),\ \ s,t = 1,\cdots,k\ . \tag{87}$$

Using Eqs.(87),(65),(69),(70) one finds the simple representation for the vector $L(x)$ (79) in the dilute gas approximation

$$L_s(x) = \frac{x-y_s}{\sqrt{\rho_0(x)}\,(|x-y_s|^2+|q_s|^2)\,|x-y_s|^2} q_s^+ u\ . \tag{88}$$

Eqs.(88),(81) give the fermionic zero modes in the dilute gas approximation. For single instanton they are reduced to the known expression [11].

Though Eq.(88) was evaluated in the dilute gas approximation, it proves to be valid for the other important case when all the instantons have identical orientation and their separations are arbitrary. This follows from the comment given after Eq.(73).

Eq.(88) may be further simplified if we are interested in the region far outside of the cores of instantons

$$|q_t|^2 \ll |x-y_t|^2,\ t=1,\cdots,k\ , \tag{89}$$

where it reads

$$L_s(x) = \frac{x-y_s}{|x-y_s|^4} q_s^+ u,\ \ s=1,\cdots,k\ , \tag{90}$$

providing the very simple expression for the fermionic zero modes (81).

Let us verify representation Eq.(78). With this purpose consider Eq.(77) from which one finds

$$\nabla_\mu D(x,y) = \frac{1}{4\pi^2}\left(-2\frac{(x-y)_\mu}{(x-y)^4}N^+(x)N(y) + \frac{1}{(x-y)^2}\nabla_\mu N^+(x)N(y)\right)\ . \tag{91}$$

This gives

$$(\gamma\nabla)D(x,y)(\gamma\nabla) = (4\pi^2)^{-1}\gamma_\mu\gamma_\nu\cdot \left[-2\frac{(x-y)_\mu}{(x-y)^4}N^+(x)N(y) + \frac{1}{(x-y)^2}\nabla_\mu N^+(x)N(y)\right]\nabla_\nu\ .$$

Here the last operator acts on the y-coordinate $\nabla_\nu = \nabla_\nu^{(y)}$. To simplify the formulae it is usufull to apply this operator to the left

$$\begin{aligned}&(\gamma\nabla)D(x,y)(\gamma\nabla) = (4\pi^2)^{-1}\gamma_\mu\gamma_\nu\cdot\\ &\Big[2\partial_\nu^{(y)}\big((x-y)_\mu/(x-y)^4\big)N^+(x)N(y) +\\ &2\big((x-y)_\mu/(x-y)^4\big)N^+(x)(\nabla_\mu^{(y)}N^+(y))^+ -\\ &\partial_\nu^{(y)}\big(1/(x-y)^2\big)\nabla_\mu N^+(x)N(y) -\\ &\big(1/(x-y)^2\big)\nabla_\mu^{(x)}N^+(x)(\nabla_\nu^{(y)}N^+(y))^+\Big]\ .\end{aligned} \tag{92}$$

The first term in the square brackets above yields $\delta(x-y)$ as it follows from the simple identity $-4\pi^2\delta(x-y) = \gamma_\mu\gamma_\nu\partial_\nu^{(y)}[2(x-y)_\mu/(x-y)^4] = \gamma_\mu\gamma_\nu\partial_\mu\partial_\nu[1/(x-y)^2]$. Then from (92) we find for the kernel $P(x,y)$ of the operator P (74)

$$\begin{aligned}&P(x,y) = [4\pi^2(x-y)^2]^{-1}[(1-\gamma_5)/2]\gamma_\mu\gamma_\nu\cdot\\ &\Big[(2/(x-y)^2)\big((x-y)_\mu N^+(x)(\nabla_\nu N^+(y))^+ -\\ &(x-y)_\nu(\nabla_\mu N^+(x))N^+(y)\big) -\\ &(\nabla_\mu N^+(x))(\nabla_\nu N^+(y))^+\Big]\ .\end{aligned} \tag{93}$$

Let us calculate the terms with derivatives in the right-hand side of (93). Note that for the considered self-dual field the potential may be presented as $A_\mu(x) = N^+(x)\partial_\mu N(x)$, see Appendix C. Therefore $\nabla_\mu N^+(x) = \partial_\mu N^+(x)(1-N(x)N^+(x))$. Using the equality $1-N(x)N^+(x) = M(x)R^{-1}(x)M^+(x)$ one finds

$$\begin{aligned}\nabla_\mu N^+(x) &= \partial_\mu N^+(x)M(x)R^{-1}(x)M^+(x) =\\ &-N^+(x)\partial_\mu M(x)R^{-1}(x)M^+(x) =\\ &N^+(x)C\tau_\mu^+R^{-1}(x)M^+(x)\ .\end{aligned} \tag{94}$$

Using the identity $M^+(x) = M^+(y) - (x-y)^+C^+$ one finds from (94)

$$\nabla_\mu N^+(x)N(y) = -N^+(x)CR^{-1}(x)\tau_\mu^+(x-y)^+C^+N(y) \tag{95}$$

because $M^+(y)N(y) = 0$. From (95) one finds

$$\begin{aligned}&(x-y)_\mu N^+(x)(\nabla_\nu N^+(y))^+ -\\ &(x-y)_\nu(\nabla_\mu N^+(x))N^+(y) =\\ &N^+(x)C\Big[(x-y)_\mu(x-y)_\nu\left(R^{-1}(x)+R^{-1}(y)\right) -\\ &i\tau^a\left[(x-y)_\mu\eta^a_{\nu\lambda}R^{-1}(y) - (x-y)_\nu\eta^a_{\mu\lambda}R^{-1}(x)\right]\cdot\\ &(x-y)_\lambda\Big]C^+N(y)\ .\end{aligned} \tag{96}$$

Here the identity $\tau^+\tau^- = \delta_{\mu\nu} + i\eta^a_{\mu\nu}\tau^a$ was used. Now we use Eq.(96) to simplify the first two terms in square brackets in (93) and Eq.(94) for the last term in (93). As a result we find

$$\begin{aligned}&P(x,y) = [4\pi^2(x-y)^2]^{-1}[(1-\gamma_5)/2]\gamma_\mu\gamma_\nu\cdot\\ &L^+(x)\Big[(1/2)(R(x)+R(y))\delta_{\mu\nu} -\\ &2i\tau^a(n_\mu n_\lambda\eta^a_{\nu\lambda}R(x) - n_\nu n_\lambda\eta^a_{\mu\lambda}R(y)) -\\ &\tau_\mu^+M^+(x)M(y)\tau_\nu^-\Big]L(y)\ .\end{aligned} \tag{97}$$

Here n_μ is a unit vector $n_\mu = (x-y)_\mu/\sqrt{(x-y)^2}$.

Now let us use the following identity for the Dirac matrixes

$$\gamma_\mu\gamma_\nu(1-\gamma_5) = (\delta_{\mu\nu} + i\eta^a_{\mu\nu}\sigma^a)(1-\gamma_5)\ . \tag{98}$$

Substituting this identity into (97) we get two terms, one comes from the $\delta_{\mu\nu}$ and the other one - from $i\eta^a_{\mu\nu}\sigma^a$ in the right-hand side of (98)

$$P(x,y) = P_{(\delta)}(x,y) + P_{(i\eta\sigma)}(x,y)\ . \tag{99}$$

For the first term we find substituting $\gamma_\mu\gamma_\nu \to \delta_{\mu\nu}$ in (97)

$$\begin{aligned}&P_{(\delta)}(x,y) = [4\pi^2(x-y)^2]^{-1}[(1-\gamma_5)/2]\cdot\\ &L^+(x)\left[2(R(x)+R(y)) - \tau_\mu^+M^+(x)M(y)\tau_\mu^-\right]L(y)\ .\end{aligned} \tag{100}$$

With the help of the identity $\tau^+_\mu q \tau^-_\mu = 4\mathrm{Re}(q)$ which is valid for any quaternion q one finds that

$$\tau^+_\mu M^+(x) M(y) \tau^-_\mu = 2[R(x) + R(y) - (x-y)^2 C^+ C] \ .$$

Then it follows from (100)

$$P_{(\delta)}(x,y) = \frac{1}{2\pi^2}\frac{1-\gamma_5}{2} L^+(x) C^+ C L(y) \ . \tag{101}$$

The second term in (99) is to be found substituting $\gamma_\mu \gamma_\nu \to i\eta^a_{\mu\nu}\sigma^a$ in (97)

$$P_{(i\eta\sigma)}(x,y) = [4\pi^2(x-y)^2]^{-1}[(1-\gamma_5)/2] \cdot$$
$$L^+(x) i\eta^a_{\mu\nu}\sigma^a\{-2i\tau^b[n_\mu n_\lambda \eta^b_{\nu\lambda} R(x) - n_\nu n_\lambda \eta^b_{\mu\lambda} R(y)] -$$
$$\tau^+_\mu M^+(x) M(y) \tau^-_\nu\} L(y) \ .$$

The first term in curly brackets is evaluated with the help of the identity $\eta^a_{\mu\nu}\eta^b_{\nu\lambda} n_\mu n_\lambda = -\delta_{ab}$. The second term is simplified with the help of $\eta^a_{\mu\nu}\tau^+_\mu q \tau^-_\nu = 4i\tau^a \mathrm{Re}(q)$ where q is an arbitrary quaternion. As a result one finds

$$P_{(i\eta\sigma)}(x,y) = -\frac{1}{2\pi^2}\frac{1-\gamma_5}{2} L^+(x) C^+ C \sigma^a \tau^a L(y) \ . \tag{102}$$

Combining (99),(101),(102) we justify Eq.(78) due to the identity $\sigma^+_\mu \tau^+_\mu = 1 - \vec{\sigma}\vec{\tau}$.

XI. APPENDIX C. THE MATRIX ELEMENT $\vec{W}_{1,2}$

In this section the explicit convenient expression for the matrix element $\vec{W}_{1,2}$ Eq.(27) is presented. Introducing convenient notation $W = i\vec{\tau}\cdot\vec{W}_{1,2}$ one can present Eq.(27) as

$$W = \frac{2}{\pi^2}\mathrm{Im}\int L^+_1(x) L_2(x) d^4x = \tag{103}$$
$$\frac{1}{\pi^2}\mathrm{Im}\int \left(L^+_1(x) L_2(x) - L^+_2(x) L_1(x)\right) d^4x \ ,$$

Here the imaginary part of a quaternion q is defined as $\mathrm{Im}\ q = (q - q^+)/2$. The second equality in Eq.(103) follows from the antisymmetry condition (28) of the matrix element $\vec{W}_{12} = -\vec{W}_{21}$. Using Eq.(79) for the functions $L_i(x),\ i = 1,2$ one finds

$$W = \frac{1}{\pi^2}\mathrm{Im}\int N^+(x) C R^{-1}(x) \Sigma R^{-1}(x) C^+ N(x) d^4x \ , \tag{104}$$

where the matrix Σ is defined as

$$\Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ . \tag{105}$$

For two instantons the matrixes $M(x), C$ in the ADHM solution have an explicit simple form

$$M(x) = \begin{pmatrix} q_1 & q_2 \\ y_1 - x & b \\ b & y_2 - x \end{pmatrix} \ , \quad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \tag{106}$$

where $q_i,\ i = 1,2$ are the quaternions describing the radius and orientation of the two instantons, y_i are the quaternions describing the positions of the instantons and

$$b = \frac{y_{12}}{2|y_{12}|^2}\left(q^+_2 q_1 - q^+_1 q_2\right) \ , \quad y_{12} = y_1 - y_2 \ . \tag{107}$$

To simplify Eq.(104) it is convenient to express the vector $N(x)$ in terms of the matrix $M(x)$. With this purpose let us use the identity

$$N(x) = a_N N_0, \quad \Pi(x) = 1 - M(x) R^{-1}(x) M^+(x), \tag{108}$$

where $\Pi(x)$ is the projection operator on the vector $N(x)$, N_0 is any vector non-orthogonal to $N(x)$: $N^+(x) N_0 \neq 0$, and a_N is the normalization coefficient which satisfies $|a_N|^2 = \left(N^+_0(1 - M(x) R^{-1}(x) M^+(x)) N_0\right)^{-1}$. It is convenient to chose

$$N^+_0 = (1, 0, 0) \ . \tag{109}$$

Substituting Eq.(108),(109) in Eq.(104), using the explicit representation (106) for the matrix $M(x)$ and the obvious identities $C^+ N_0 = 0$ and $R^{-1}(x) \Sigma R^{-1}(x) = \Sigma / \det R(x)$ one finds

$$W = \frac{1}{\pi^2}\mathrm{Im}\int \frac{Q R^{-1}(x) m^+(x) \Sigma m(x) R^{-1}(x) Q^+}{\det R(x)\,(1 - Q R^{-1}(x) Q^+)}\, d^4x \ . \tag{110}$$

Here $m(x)$ is "the square part" of $M(x)$:

$$m(x) = \begin{pmatrix} y_1 - x & b \\ b & y_2 - x \end{pmatrix} \ , \tag{111}$$

and Q, Q^+ are the two-dimensional quaternions

$$Q = (q_1, q_2) \ , \quad Q^+ = \begin{pmatrix} q^+_1 \\ q^+_2 \end{pmatrix} \ . \tag{112}$$

Representation (110) is convenient for direct numerical calculations as well as for asymptotic expansion in different regions. The most simple and interesting is the case of large separation, $r_{12} = |y_{12}| \gg |q_1|, |q_2|$. For this case the main contribution to the integral in Eq.(110) gives the region $|y_1 - x| \sim |y_2 - x| \sim r_{12}$ in which

$$m(x) \approx m_0(x) = \begin{pmatrix} y_1 - x & 0 \\ 0 & y_2 - x \end{pmatrix} \ , \tag{113}$$

and

$$R(x) \approx \begin{pmatrix} |y_1 - x|^2 & 0 \\ 0 & |y_2 - x|^2 \end{pmatrix} \ . \tag{114}$$

Substituting Eqs.(113),(114) in Eq.(110) and taking into account that $|Q R^{-1}(x) Q^+| \ll 1$ one finds

$$W = 2\frac{\mathrm{Im}(q_1 q^+_2)}{r^2_{12}} \ , \tag{115}$$

which agrees with Eq.(37) evaluated in Section IV using the different technic.

The other important case provides the region $r^2_{12} \gg |q_1 q_2 \sin\gamma_{12}|$ in which the instanton separation might be not only large but small as well, less then the instantons radiuses, provided their orientations are close $|\gamma_{12}| \ll 1$. In this case b in Eq.(111) is smaller then the separation $|b| \ll r_{12}$. Therefore one can again use the asymptotic Eq.(113) for $m(x)$. In contrast the matrix $R(x)$ is to be considered more accurately,

$$R(x)_{ij} \approx q_i^+ q_j + \left(m_0^+(x) m_0(x)\right)_{ij} \ , \quad i,j = 1,2 \ , \quad (116)$$

because the separation might be less then the radiuses. Substituting (113),(116) in Eq.(110) one finds after simple algebraic calculations

$$W = 2F(\kappa_1, \kappa_2) \frac{\mathrm{Im}(q_1 q_2^+)}{r_{12}^2} \ , \quad (117)$$

where $\kappa_i = 2|q_i|/r, \ i = 1,2$ and the function $F(\kappa_1, \kappa_2)$ is

$$F(\kappa_1, \kappa_2) = \frac{4}{\pi^2} \int \left(|x|^2 - 1\right) \times \quad (118)$$
$$\frac{d^4x}{|x-1|^2|x+1|^2\left[\left(|x-1|^2+\kappa_1^2\right)\left(|x+1|^2+\kappa_2^2\right)-\kappa_1^2\kappa_2^2\right]} \ .$$

The explicit form of this integral representation for $F(\kappa_1, \kappa_2)$ is evaluated in the convenient coordinate frame $y_1 = -y_2 = 1$. The integral in Eq.(118) is easily reduced to the one-dimensional one. Its properties are discussed in detail is Section V.

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[1] M.Yu.Kuchiev, Europhys.Lett. **28**, 539 (1994).

[2] L. Brink, M. Henneaux, *Principles of String Theory* (Plenum Press, New York and London, 1988).

[3] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu.S.Tyupkin, Phys. Lett. **59B**, 85 (1975).

[4] S. Elitzur, Phys. Rev. **D12**, 3978 (1975).

[5] M. Creutz, *Qurks, gluons and lattices* (Cambridge University Press, Cambridge, London, NY, New Rochelle, Melbourne, Sydney) (1983).

[6] A. M. Polyakov, *Gauge fields and strings* (Harwood Academic Publishers, Chur, London, Paris, NY, Mehlbourn) (1987).

[7] L. S. Brown, R. D. Carlitz, D. B. Creamer, and C. Lee, Phys. Rev. D **17**, 1583 (1978).

[8] N. H. Christ, E. J. Weinberg, and N.K.Stanton, Phys. Rev. D **18**, 2013 (1978).

[9] C. Callan, K. Dashen, and D. Gross, Phys. Rev. D **17**, 2717 (1978); *ibid* **19**, 1826 (1979)

[10] M. F. Atiyah, N. J. Hitchin, V. G. Drinfield, and Yu. I. Manin, Phys. Lett . **A65**, 185 (1978).

[11] G. 't Hooft, Phys. Rev. D **14**, 3432 (1976).

[12] M. Yu.Kuchiev, Europhys.Lett. **30**, 225 (1995).

[13] M. F. Atiyah, I.M.Singer, Ann.Math. **87** 484 (1968).

[14] J. Kiskis, Phys.Rev. D **15**, 2329 (1977).

[15] A. I. Vainshtein, V. I. Zakharov, V. A. Novikov, and M. A. Shifman, Usp. Fiz. Nauk **136**, 553 (1982).

[16] P. W. Higgs, Phys.Rev. **145**, 1156 (1966).

[17] E. Corrigan, D. Fairlie, Phys.Lett. **67B**, 69 (1977).

[18] G. t'Hooft, (unpublished, see reference in [20]).

[19] F. Wilczek, in:*Quark Confinement and Field Theory* eds. D. Stump and D. Weingarten, (Wiley, New York), (1977).

[20] M. K. Prasad, Physica **1D** 167 (1980).

Fig.1

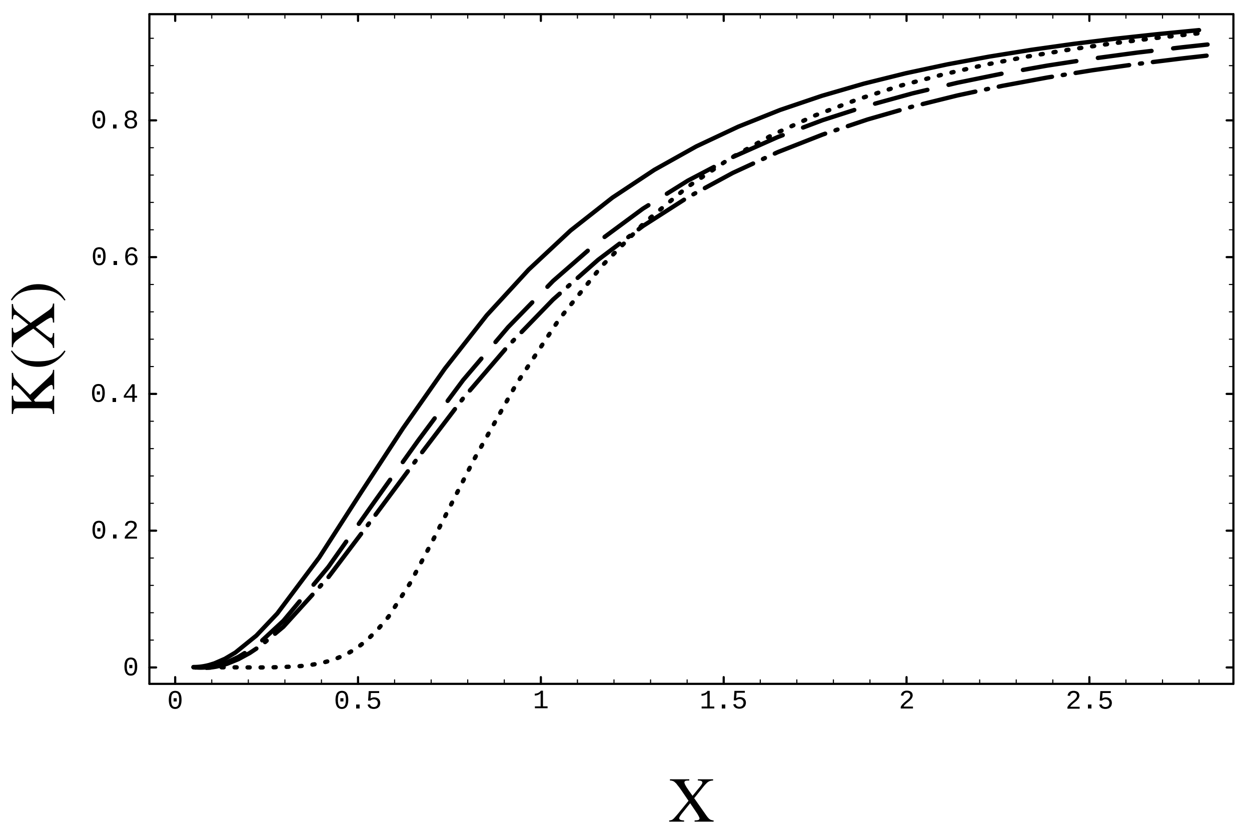

FIG. 1. The interaction of instantons for small separation. The coefficient $K(X)$ describes how strongly the interaction of instantons deviates from the asymptotic value, see Eq.(43), is shown vs the relative separation between instantons $X = r_{12}/(\rho_1+\rho_2)$. The different curves present the dependence of $K(X)$ on the ratio of instanton radiuses ρ_1/ρ_2 and on the angle γ_{12} between the directions of the instanton orientations, $\cos\gamma_{12} = n_{1,\mu}n_{2,\mu}$. The full curve: $\rho_1/\rho_2 = 1$, $\gamma_{12} = 0$, the dashed curve: $\rho_1/\rho_2 = 5$, $\gamma_{12} = 0$, the dashed-dotted curve: $\rho_1/\rho_2 = 10$, $\gamma_{12} = 0$, the dotted curve: $\rho_1/\rho_2 = 1$, $\gamma_{12} = \pi/2$. The coefficient $K(X)$ is found as $K(X) = F^2(\kappa_1, \kappa_2)$, see Eq.(118), for $\gamma_{12} = 0$. For calculations with $\gamma_{12} = \pi/2$ the more general Eq.(110) is used. The drop in $K(X)$ for small separation indicates that in this region the instanton interaction is suppressed.

Fig.2

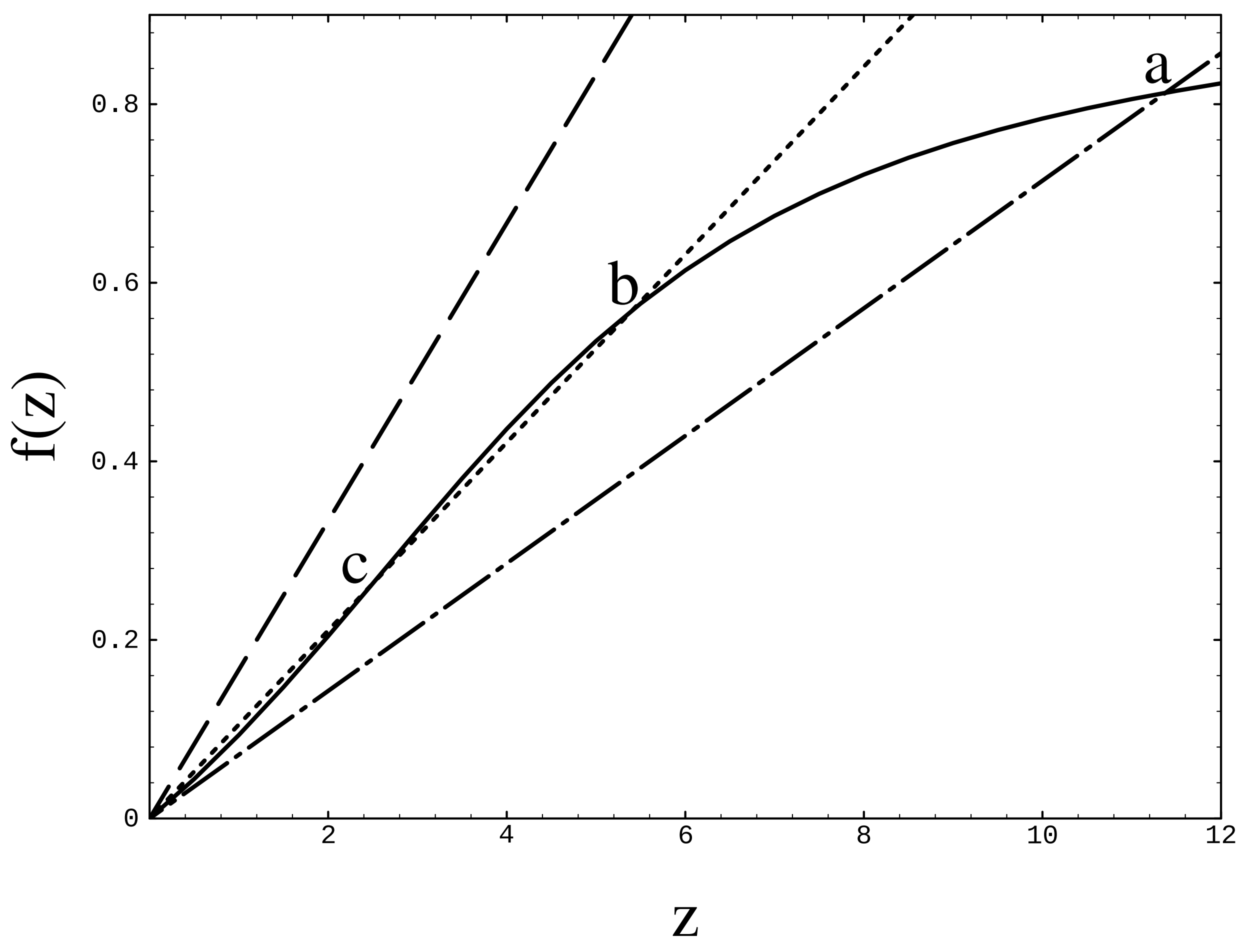

FIG. 2. The mean-field approximation. The full curve: the right-hand side of Eq.(56) defined in Eq.(57). The straight lines give the left-hand side of Eq.(56) for different parameters B. The dashed line: $B < B_c$, the dotted line: $B_c < B < B'_c$, the dashed-dotted curve: $B'_c < B$, where $B_c = 1.14,\ B'_c = 3/2$ and the three considered values of B are $B = 0.75,\ 1.19,\ 1.75$. The points "a,b,c" show the nontrivial solutions in which the instantons are polarized. The point "a" indicates the unique nontrivial solution exiting for large B. The points "b,c" are the two solutions existing for intermediate B. One of the solutions, the point "b", is stable, the other solution, the point "c", is non-stable to the fluctuations of the instanton polarization. The mean field approximation confirms the possibility of the state with polarized instantons.